Statistics of largest cluster growth through constant rate random filling of lattices

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In this paper we consider a percolation model where the probability p for a site to be occupied increases linearly in time, from 0 to 1. We analyze the way the largest cluster grows in time, and in particular, we study the statistics of the ''jumps'' in the mass of the largest cluster, and of the time delay between those events. Different critical behaviors are observed below and above the percolation threshold. We propose a theoretical analysis, and we check our results against direct numerical simulations.

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I. INTRODUCTION

Percolation is now a well-known theory that has been applied successfully to a wide variety of problems. Most critical properties of this model have been clarified through a considerable number of published works. Many excellent reviews on this theory can be found in Refs. [1–6].

However, most studies either concerned the static properties at the critical point, or time-dependent properties of variants of the standard percolation problem since no time is present in the latter. These variants (such as ''spreading percolation'' [7], ''invasion percolation'' [8], ''stirred percolation'' [9], ''self-organized percolation'' [10], ''kinetic growth models'' [11]. etc.), which include time, generally also affect the generic definition of the percolation model. Albeit most often scaling relations can be derived to relate the dynamical properties of these models to the critical exponents of percolation, the connection is indirect.

In the present study, we consider the standard percolation model, with an occupation probability p continuously increasing in time from 0 to 1. In practice, a new empty site will be added (''occupied'') at every time step in a lattice of finite size L, so that $p=t/L^d$. We specifically study the evolution of the largest cluster $M(t)$ at each time step. Its mass does not increase in a steady fashion, but rather by sudden increases [hereafter referred to as ''jumps'' $j=M(t+1)-M(t)$] separated by long time intervals (called ''delays'' hereafter) where the largest cluster remains unchanged ($j=0$). Those jumps and delays do vary considerably as p goes from 0 to 1. However, as will be shown below, if the distributions of jumps and delays are integrated over the entire range of p values, those distributions will capture some aspects of the critical behavior encountered in the vicinity of the percolation threshold. Since fluctuations are singular at the critical point, the above quantities receive dominant contributions from the vicinity of the percolation threshold, and thus the statistics of the jumps preserve information from the critical point although the latter has been swept through in this process. Indeed the statistical distribution of jump sizes has a simple power-law behavior. Surprisingly, if we stop at the percolation threshold, the jump size distribution integrated from $p=0$ to $p=p_c$ also displays a power law form, but with a *different* exponent. Concerning the delay between jumps we show that their distribution consists not in one, but *two* power laws. These predictions are checked against numerical simulations.

The motivation for this work is twofold. First, there are systems in which the percolation control parameter can be continuously adjusted, such as composites in which the temperature allows to open or close contacts between particles [12,13]. In such systems, one may be interested in quantifying the fluctuations in any macroscopic property sensitive to the connectivity (such as the electrical conductivity). Similar attempts have been proposed in a related context such as the study of resistance jumps during mercury porosimetry at the onset of the breakthrough point, by Katz *et al.* [14] (the latter problem was revisited theoretically in Refs. [15,16]). In those problems, it may be difficult to adjust the control parameter to stop the system precisely at its percolation threshold. However, a more secure procedure would be to vary continuously the control parameter in order to scan through the percolation. We have chosen such a procedure here. The mass of the largest cluster is only one among the many observables that can be chosen, but it is the most natural critical quantity to consider in the context of percolation since it concerns the order parameter, and as mentioned above, it already displays surprising features. The second motivation comes from algorithms recently proposed to locate the critical point. The principle of such methods is to either increase or decrease the control parameter, depending on whether the order parameter is smaller or larger than a prescribed value. The convergence of the method and the oscillation of the control parameter around the critical point are controlled by the ''noise'' (both jumps and delays to use the above terminology). These two applications are not specifically studied in the present paper, but they both require the results presented herewith.

We use here the standard notations used in percolation theory. p denotes the occupation probability, and p_c the percolation threshold. We also call ϵ the difference $\epsilon\equiv|p-p_c|$. L is the system size. Classically, the correlation length ξ and probability to belong to the infinite cluster, P_∞, have a critical behavior characterized by the exponents ν and β, such that $\xi \propto |\epsilon|^{-\nu}$ and $P_\infty \propto \epsilon^\beta$ above threshold. D will denote the fractal dimension of the infinite cluster at the percolation threshold. It is related to the previous exponents through $D = d - \beta/\nu$.

We will first consider the time intervals between jumps, considering the p values above percolation below the threshold, and then below threshold. Then we will study the distribution of the jump amplitude, above and below the percolation threshold.

II. DISTRIBUTION OF TIME INTERVALS BETWEEN JUMPS ABOVE PERCOLATION THRESHOLD

Let us consider the limit of a large system size so that the addition of a few sites does not affect the value of p. Then at each p, the probability distribution $\varphi_1(t,\epsilon)$ of time intervals t, between jumps is an exponential distribution (Poisson process) of characteristic time $T \propto 1/P_{\text{jump}}$ where P_{jump} is the probability that the newly added site is the nearest neighbor to the largest cluster. The number of those vacant sites is proportional to the mass of the largest cluster itself. For p larger than the percolation threshold, P_{jump} approaches 1 and hence the time interval between jumps T will be close to 1. In the vicinity of the percolation threshold, P_{jump} vanishes and using the critical behavior of the mass of the largest cluster, we can write

$$P_{\text{jump}} \propto \xi^{D-d} = \epsilon^\beta. \tag{1}$$

Exploiting the definition of the Poisson process, in the vicinity of the percolation threshold, we can write

$$\varphi_1(t,\epsilon) = (1/T)\exp(-t/T) = A\epsilon^\beta \exp(-At\epsilon^\beta), \tag{2}$$

where A is a constant.

The statistical distribution $\varphi_1(t)$ of t as obtained by sweeping through p values continuously from p_c to 1, is obtained by superimposing the above $\varphi_1(t,\epsilon)$, with a weight proportional to P_{jump} and this for all values of ϵ,

$$\varphi_1(t) = \int_0^{(1-p_c)} \varphi_1(t,\epsilon) P_{\text{jump}}(\epsilon) d\epsilon$$
$$\propto t^{-2-1/\beta} \left\{ \int_0^\infty \exp(-u^\beta) u^{2\beta} du \right\}, \tag{3}$$

hence $\varphi_1(t) \propto t^{-(2+1/\beta)}$, i.e., a very rapid decay since $2 + 1/\beta \approx 9.2$ in two dimensions. The above power-law derivation has been obtained using the critical behavior of P_{jump} and thus it is only valid for large t. However, the decay of T with p is so fast that the critical behavior dominates the entire distribution.

The range of time scales $[T_0, T_1]$ covered by this regime is expected to be small, considering the large value of this exponent. Indeed, the smallest time is encountered for p close to 1, and is of the order of 1, $T_0 \propto L^0$. The longest times are observed in the vicinity of the threshold where $T_1 \propto L^{d-D}$ and $D = d - \beta/\nu$ is the fractal dimension of the infinite cluster at percolation threshold.

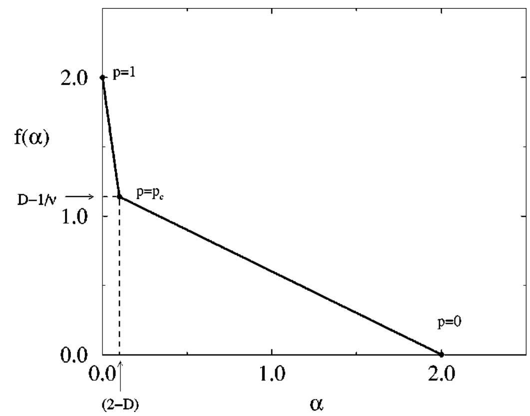

FIG. 1. Spectrum of the delay between jumps in the $f(\alpha)$ formalism. The first steep slope describes the jump distribution above percolation threshold, whereas the second one is relative to the jumps occurring below threshold.

The above scaling law is mostly of theoretical interest, since observing this power-law decay over one decade in size requires in two dimensions systems of size $L \approx 10^{10}$, a gigantic number. However, what is expected for more reasonnable system sizes is an abrupt decay for small time intervals.

Multifractal representation

Obviously, the number of jumps and the time delay between jumps will scale as power laws of the system size. To single out exponents, the natural language is the so-called multifractal framework. Here the word multifractal is not quite appropriate since very few dimensions will appear, however we borrow from this framework the quantities α and f, conventionally defined from the number of delays $N(t) d\ln(t)$ in the range $[t, t(1 + d\ln(t))]$ using a logarithmic measure for the time, hence $N(t) = L^2 \varphi_1(t) t$,

$$\alpha \equiv \lim_{L\to\infty} \frac{\ln(t)}{\ln(L)},$$

$$f(\alpha) \equiv \lim_{L\to\infty} \frac{\ln[N(t)]}{\ln(L)}. \tag{4}$$

The function $f(\alpha)$ gives a spectrum that is representative for the scaling properties of the distribution.

We have just seen that the range of α values covered by this regime is from $\alpha_0 = 0$ to $\alpha_1 = 2 - D$. Let us now consider the corresponding values of f.

In terms of the number of jumps, in two dimensions, we have of order $N_0 \propto L^2$ jumps of size T_0, hence $f(\alpha_0) = 2$. At threshold, the number density of jumps is $N_1 \propto L^2 \varphi_1(T_1) T_1 \propto L^{D-1/\nu}$. Hence $f(\alpha_1) = D - 1/\nu \approx 1.14$. In between these two points, $(\alpha_0, f(\alpha_0))$ and $(\alpha_1, f(\alpha_1))$, it is a simple matter to show that the power-law distribution φ_1 will give rise to a straight line connecting the two points. Figure 1 shows the spectrum in which we have indicated the values of p that contribute dominantly.

III. DISTRIBUTION OF TIME INTERVALS BETWEEN JUMPS BELOW THRESHOLD

Let us now consider jumps occuring below threshold. Here the situation is a little different since the largest cluster will change from time to time. This makes the situation apparently complicated since one has to consider cases where two large clusters smaller than the largest one, merge to become the largest. The exact formulation of this event involves quantities that are difficult to evaluate. However, reversing the arrow of time simplifies the problem considerably. As time decreases, the occurence of a jump means that a site has been chosen (and removed) from the largest cluster, with no other conditions. The question is now to estimate the probability of this event.

Again, we focus first on the vicinity of the percolation threshold. The largest cluster has a size $s^* \propto \epsilon^{-1/\sigma}$ where $\sigma \equiv 1/(\nu D)$. The total number of occupied sites is pL^d (in the neighborhood of p_c, the latter can be considered as independent of p and equal to $p_c L^d$). Thus the probability for a jump is

$$P_{\text{jump}} \propto L^{-d} \epsilon^{-1/\sigma}. \tag{5}$$

We can now reproduce the same argument as previously. The statistical distribution φ_2 of delay between jumps is still a Poisson distribution with a characteristic time $T \propto P_{\text{jump}}^{-1}$. So that integrating over p values from 0 up to p_c

$$\varphi_2(t) \propto \int_0^{p_c} \exp(-A' L^{-d} t \epsilon^{-1/\sigma}) L^{-2d} \epsilon^{-2/\sigma} d\epsilon$$
$$\propto t^{-2+\sigma} L^{-d\sigma} \int_0^{p_c/2} \exp(-u^{-1/\sigma}) u^{-2/\sigma} du. \tag{6}$$

Hence $\varphi_2(t) \propto t^{-2+\sigma}$ or numerically in two dimensions $2-\sigma \approx 1.6$. Again, here we have implicitly extended the critical behavior down to $p=0$. This is legitimate only for times much smaller than the ones occuring close to $p=0$. The latter regime that give rise to the largest times, corresponds also to very rare events and thus the above distribution is dominated by the critical behavior over most of its observable range of time intervals.

Multifractal representation

What is the range of time delays $[T_1, T_2]$ covered by this scaling law? The largest time delays are those found for small p. In this case, the typical time interval is $T_2 \propto L^2$ in two dimensions. The smallest comes from the vicinity of p_c, and is obviously limited by finite-size effects. The lower time constant obtained here is of the order of $T_1 \propto L^{2-D}$, which matches the longest time delay of the above-threshold regime. Thus this regime is much wider than the previous one, although it contains much less events. Therefore, the α parameter here varies from $2-D$ to 2.

At these times, we can compute the number of jumps as defined in the preceding section. N_1 is identical to the expression computed above and $N_2 \propto L^0$, hence providing the final point of the spectrum, $\alpha_2 = 2$ and $f(\alpha_2) = 0$. We have obtained the complete spectrum shown in Fig. 1.

IV. DISTRIBUTION OF JUMPS ABOVE THRESHOLD

The size of the jumps is related to the cluster size distribution, since removing a site from the infinite cluster produces a finite cluster whose size is $j-1$ where j is the ‘‘jump.’’ However, the statistical distribution of jump is not identical to the cluster size distribution because the probability to disconnect a cluster of size s depends on its morphology and not only its mass, and hence this bias the jump size distribution.

Let us assume that at a fixed value of p, the jump size distribution, $n(j,p)$, consists in a power-law of (by now unknown) exponent τ', truncated in the same way as the cluster size distribution (the maximum cluster size is also the maximum jump size). Thus we write the jump size distribution as

$$n(j,\epsilon) = j^{-\tau'} \psi(j \epsilon^{1/\sigma}), \tag{7}$$

where $\psi(x)$ is a constant for small arguments $x \ll 1$, and decays rapidly to zero for $x \gg 1$.

When the jumps are integrated all along the process with p varying from p_c to 1, the resulting distribution $n_1(j)$ is obtained from

$$n_1(j) = \int_0^{1-p_c} n(j,\epsilon) P_{\text{jump}}(\epsilon) d\epsilon$$
$$\propto j^{-\tau'} \int_0^{1-p_c} \psi(j\epsilon^{1/\sigma}) \epsilon^{\beta} d\epsilon \tag{8}$$
$$\propto j^{-\tau' - (1+\beta)\sigma}.$$

Hence, it is a power-law distribution $n_1(j) \propto j^{-\tau''}$ with

$$\tau'' = \tau' - 1 + \frac{1}{D}\left(\frac{1}{\nu} + 2\right). \tag{9}$$

In order to determine τ'' and hence τ', we propose to compute the first moment of the jump size distribution $J = L^d \int n_1(j) j dj$. The latter should be equal to $L^d - M_\infty$, where M_∞ is the mass of the infinite cluster at threshold $M_\infty \propto L^D$. From the distribution of j, we get

$$J \propto L^d \int_1^{L^D} j^{1-\tau''} dj. \tag{10}$$

Thus identification with the previous expression leads to

$$D = d + (2 - \tau'')D \tag{11}$$

or

$$\tau'' = 1 + \frac{d}{D} = \tau. \tag{12}$$

Therefore, the *integrated* jump size distribution n_1 from p_c to 1 has exactly the same exponent as the cluster size distri-

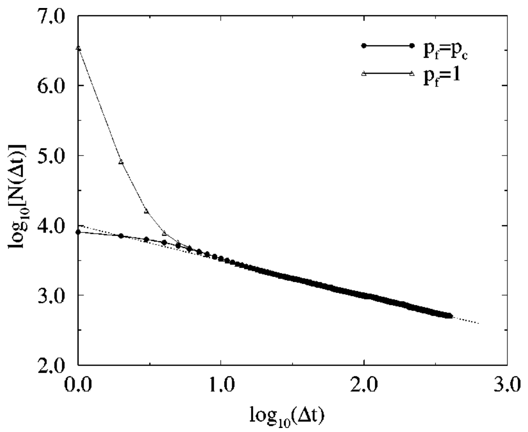

FIG. 2. Log-log plot of the histogram of time intervals between jumps. The histogram is obtained for p values integrated in the the interval $[0,p_f]$, with $p_f=p_c$ (symbol ●) and $p_f=1$ (symbol △). The dotted line is a linear regression through the data. The system size is $L=3000$.

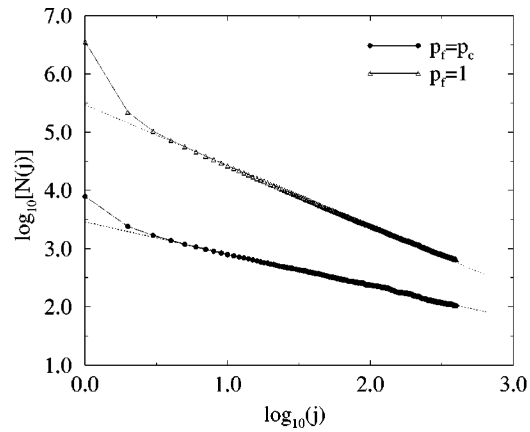

FIG. 3. Log-log plot of the histogram of jump sizes. The histogram is obtained for p values integrated in the interval $[0,p_f]$, with $p_f=p_c$ (symbol ●) and $p_f=1$ (symbol △). The dotted lines are linear regressions through the data. The system size is $L=3000$.

bution at threshold. This result, in spite of its apparent simplicity is not obvious since, at threshold, the jump size distribution $n(j,p_c)$ has *a different exponent*, τ'. From the previous equations we can now obtain its expression as

$$\tau'=\tau+1-\frac{1}{D}\left(\frac{1}{\nu}+2\right)=2-\sigma+\frac{(d-2)}{D} \tag{13}$$

or $\tau'\approx 1.6$ in two dimensions, as compared to $\tau\approx 2.05$.

V. DISTRIBUTION OF JUMPS BELOW THRESHOLD

Below the threshold, we have to integrate the distribution of jumps, over all p values, however, at the threshold, the frequency of jumps is much larger than for small p values, and the distribution of jumps contains large as well as small values at threshold. For small p the jumps are only small ones, and rare. Hence, here the immediate vicinity of thresold will mask contributions from small p values, and hence,

$$n_2(j)\propto n_1(j,p_c) \tag{14}$$

or

$$n_2(j)\propto j^{-\tau'}\propto j^{-2+\sigma}. \tag{15}$$

However, in contrast with the case of the distribution of delays, j first increases and reaches its maximum $\propto L^D$ at $p=p_c$, and then decreases back to 1. The distribution of all jump sizes from $p=0$ up to threshold n_2, is progressively erased as p increases from p_c to 1, and is replaced by n_1. No memory is preserved from the regime $p<p_c$ because the total number of jumps is much greater above threshold.

VI. NUMERICAL SIMULATIONS

We have performed numerical simulation of this problem keeping track of the clusters, as sites are progressively occupied on a square lattice. System size up to $L=4000$ have been considered. Figure 2 shows the log-log plot of the cumulative number of time intervals between jumps, $N(t)=\int_t^\infty n(t')dt'$, occurring between $p=0$ and p_f, for two different choices of p_f, either p_c or 1. A good agreement with the predicted behavior is observed. A first power-law distribution is obtained at $p_f=p_c$, with a measured exponent 1.5 to be compared with the prediction $2-\sigma=1.6$. Above threshold, all time intervals t are extremely small (less than about 5), and the distribution develops a very abrupt peak at $t=1$. As discussed above, the predicted slope $1+1/\beta\approx 8.2$ of this regime cannot be measured in practice.

Figure 3 shows the cumulative histogram of the jump sizes for all jumps taking place from $p=0$ to p_f, for both values of $p_f=p_c$ and $p_f=1$. Both plots show a power-law behavior, with exponents, respectively, $\tau'=1.55$ and $\tau''=2.04$, to be compared to the predicted values $\tau'=1.60$ and $\tau''=2.05$. Thus, both data sets concerning the jump size distribution and the delay between jumps are fully consistent with our predictions.

VII. CONCLUSIONS

Following the same lines, previous results can be extended to other physical quantities displaying a critical behavior at the percolation threshold, (e.g., discontinuities of the conductance). Thus, this may allow to have access to the statistics of fluctuations as the p parameter is continuously varied through the critical point. In particular, it has been shown in various composite systems that temperature could play the role of p [12,13]. This was proposed to give quantitative account of the variation of the conductivity of dirty superconductors or of epoxy/carbon black composite with temperature. The unusual noise properties of these systems might be addressed using the present approach.

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